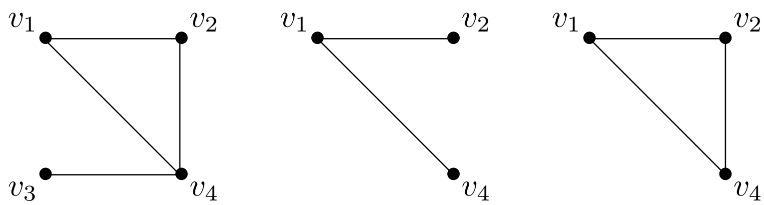

Figure 5.1.5 Left to right: a graph, a subgraph, an induced subgraph.

If a graph G is not connected, define $v \sim w$ if and only if there is a path connecting v and w. It is not hard to see that this is an equivalence relation. Each equivalence class corresponds to an induced subgraph G; these subgraphs are called the **connected components** of the graph.

Exercises 5.1.

1. Prove that if $\sum_{i=1}^{n} d_i$ is even, there is a graph (not necessarily simple) with degree sequence $d_1, d_2, \ldots, d_n$.
2. Prove that $0, 1, 2, 3, 4$ is not graphical.
3. Is $4, 4, 3, 2, 2, 1, 1$ graphical? If not, explain why; if so, find a simple graph with this degree sequence.
4. Is $4, 4, 4, 2, 2$ graphical? If not, explain why, and find a graph with this degree sequence; if so, find a simple graph with this degree sequence.
5. Prove that a simple graph with $n \geq 2$ vertices has two vertices of the same degree.
6. Prove the "only if" part of theorem 5.1.3.
7. Draw the 11 non-isomorphic graphs with four vertices.
8. Suppose $G_1 \cong G_2$. Show that if G_1 contains a cycle of length k so does G_2.
9. Define $v \sim w$ if and only if there is a path connecting vertices v and w. Prove that $\sim$ is an equivalence relation.

5.2 Euler Cycles and Paths

The first problem in graph theory dates to 1735, and is called the Seven Bridges of Königsberg. In Königsberg were two islands, connected to each other and the mainland by seven bridges, as shown in figure 5.2.1. The question, which made its way to Euler, was whether it was possible to take a walk and cross over each bridge exactly once; Euler showed that it is not possible.

We can represent this problem as a graph, as in figure 5.2.2.

The two sides of the river are represented by the top and bottom vertices, and the islands by the middle two vertices. There are two possible interpretations of the question, depending on whether the goal is to end the walk at its starting point. Perhaps inspired by this problem, a **walk** in a graph is defined as follows.

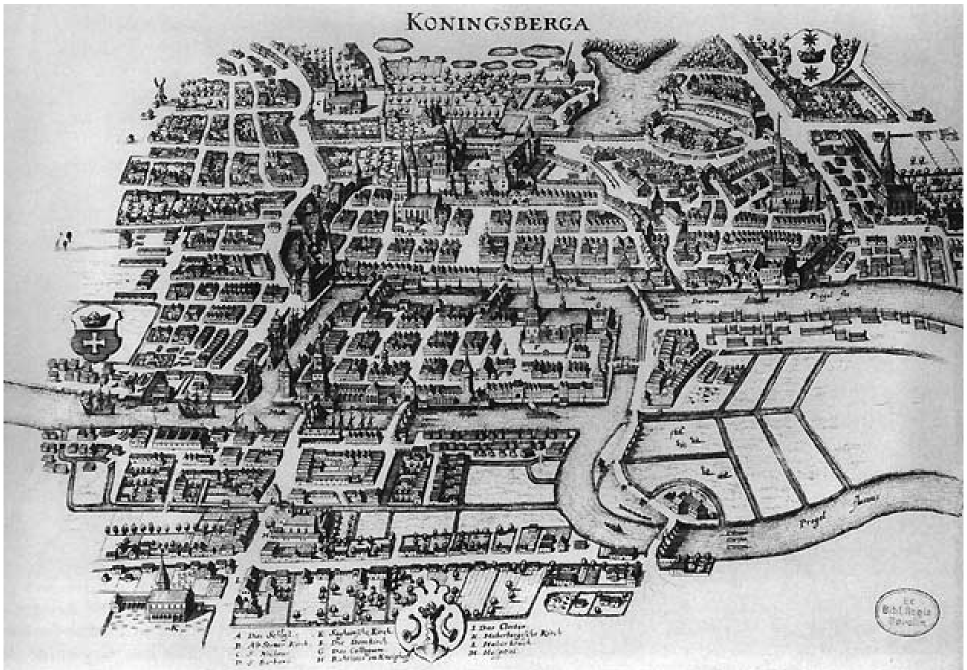

Figure 5.2.1 The Seven Bridges of Königsberg.

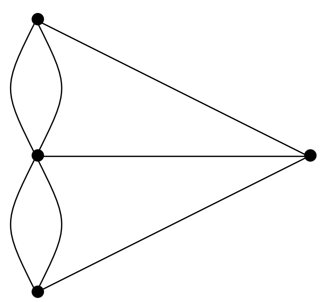

Figure 5.2.2 The Seven Bridges of Königsberg as a graph.

DEFINITION 5.2.1 A walk in a graph is a sequence of vertices and edges,

$$v_1, e_1, v_2, e_2, \ldots, v_k, e_k, v_{k+1}$$

such that the endpoints of edge e_i are v_i and v_{i+1}. In general, the edges and vertices may appear in the sequence more than once. If $v_1 = v_{k+1}$, the walk is a **closed walk**. □

We will deal first with the case in which the walk is to start and end at the same place. A successful walk in Königsberg corresponds to a closed walk in the graph in which every edge is used exactly once.

What can we say about this walk in the graph, or indeed a closed walk in any graph that uses every edge exactly once? Such a walk is called an **Euler cycle**. Certainly the graph must be connected, as this one is. Beyond that, imagine tracing out the vertices and edges of the walk on the graph. At every vertex other than the common starting and ending point, we come into the vertex along one edge and go out along another; this can happen more than once, but since we cannot use edges more than once, the number of edges incident at such a vertex must be even. Already we see that we're in trouble in this particular graph, but let's continue the analysis. The common starting and ending point may be visited more than once; except for the very first time we leave the starting vertex, and the last time we arrive at the vertex, each such visit uses exactly two edges. Together with the edges used first and last, this means that the starting vertex must also have even degree. Thus, since the Königsberg Bridges graph has odd degrees, the desired walk does not exist.

The question that should immediately spring to mind is this: if a graph is connected and the degree of every vertex is even, is there an Euler cycle? The answer is yes.

THEOREM 5.2.2 If G is a connected graph, then G contains an Euler cycle if and only if every vertex has even degree.

Proof. We have already shown that if there is an Euler cycle, all degrees are even.

We prove the other direction by induction on the number of edges. If G has no edges the problem is trivial, so we assume that G has edges.

We start by finding some closed walk that does not use any edge more than once: Start at any vertex v_0; follow any edge from this vertex, and continue to do this at each new vertex, that is, upon reaching a vertex, choose some unused edge leading to another vertex. Since every vertex has even degree, it is always possible to leave a vertex at which we arrive, until we return to the starting vertex, and every edge incident with the starting vertex has been used. The sequence of vertices and edges formed in this way is a closed walk; if it uses every edge, we are done.

Otherwise, form graph G' by removing all the edges of the walk. G' is not connected, since vertex v_0 is not incident with any remaining edge. The rest of the graph, that is, G' without v_0, may or may not be connected. It consists of one or more connected subgraphs, each with fewer edges than G; call these graphs G_1, G_2,…,G_k. Note that when we remove the edges of the initial walk, we reduce the degree of every vertex by an even number, so all the vertices of each graph G_i have even degree. By the induction hypothesis, each G_i has an Euler cycle. These closed walks together with the original closed walk use every edge of G exactly once.

Suppose the original closed walk is $v_0, v_1, \ldots, v_m = v_0$, abbreviated to leave out the edges. Because G is connected, at least one vertex in each G_i appears in this sequence, say vertices $w_{1,1} \in G_1$, $w_{2,1} \in G_2$,…, $w_{k,1} \in G_k$, listed in the order they appear in

$v_0, v_1, \ldots, v_m$. The Euler cycles of the graphs G_i are

$$\begin{aligned}
&w_{1,1}, w_{1,2}, \ldots, w_{1,m_1} = w_{1,1}\\
&w_{2,1}, w_{2,2}, \ldots, w_{2,m_2} = w_{2,1}\\
&\vdots\\
&w_{k,1}, w_{k,2}, \ldots, w_{k,m_k} = w_{k,1}.
\end{aligned}$$

By pasting together the original closed walk with these, we form a closed walk in G that uses every edge exactly once:

$$\begin{aligned}
v_0, v_1, \ldots, v_{i_1} &= w_{1,1}, w_{1,2}, \ldots, w_{1,m_1} = v_{i_1}, v_{i_1+1},\\
\ldots, v_{i_2} &= w_{2,1}, \ldots, w_{2,m_2} = v_{i_2}, v_{i_2+1},\\
\ldots, v_{i_k} &= w_{k,1}, \ldots, w_{k,m_k} = v_{i_k}, v_{i_k+1}, \ldots, v_m = v_0.
\end{aligned}$$

■

Now let's turn to the second interpretation of the problem: is it possible to walk over all the bridges exactly once, if the starting and ending points need not be the same? In a graph G, a walk that uses all of the edges but is not an Euler cycle is called an **Euler path**. It is not too difficult to do an analysis much like the one for Euler cycles, but it is even easier to use the Euler cycle result itself to characterize Euler paths.

THEOREM 5.2.3 A connected graph G has an Euler path if and only if exactly two vertices have odd degree.

Proof. Suppose first that G has an Euler path starting at vertex v and ending at vertex w. Add a new edge to the graph with endpoints v and w, forming G'. G' has an Euler cycle, and so by the previous theorem every vertex has even degree. The degrees of v and w in G are therefore odd, while all others are even.

Now suppose that the degrees of v and w in G are odd, while all other vertices have even degree. Add a new edge e to the graph with endpoints v and w, forming G'. Every vertex in G' has even degree, so by the previous theorem there is an Euler cycle which we can write as

$$v, e_1, v_2, e_2, \ldots, w, e, v,$$

so that

$$v, e_1, v_2, e_2, \ldots, w$$

is an Euler path. ■

Exercises 5.2.

1. Suppose a connected graph G has degree sequence $d_1, d_2, \ldots, d_n$. How many edges must be added to G so that the resulting graph has an Euler cycle? Explain.
2. Which complete graphs K_n, $n \geq 2$, have Euler cycles? Which have Euler paths? Justify your answers.
3. Prove that if vertices v and w are joined by a walk they are joined by a path.
4. Show that if G is connected and has exactly $2k$ vertices of odd degree, $k \geq 1$, its edges can be partitioned into k walks. Is this true for non-connected G?

5.3 Hamilton Cycles and Paths

Here is a problem similar to the Königsberg Bridges problem: suppose a number of cities are connected by a network of roads. Is it possible to visit all the cities exactly once, without traveling any road twice? We assume that these roads do not intersect except at the cities. Again there are two versions of this problem, depending on whether we want to end at the same city in which we started.

This problem can be represented by a graph: the vertices represent cities, the edges represent the roads. We want to know if this graph has a cycle, or path, that uses every vertex exactly once. (Recall that a cycle in a graph is a subgraph that is a cycle, and a path is a subgraph that is a path.) There is no benefit or drawback to loops and multiple edges in this context: loops can never be used in a Hamilton cycle or path (except in the trivial case of a graph with a single vertex), and at most one of the edges between two vertices can be used. So we assume for this discussion that all graphs are simple.

DEFINITION 5.3.1 A cycle that uses every vertex in a graph exactly once is called a **Hamilton cycle**, and a path that uses every vertex in a graph exactly once is called a **Hamilton path**. □

Unfortunately, this problem is much more difficult than the corresponding Euler cycle and path problems; there is no good characterization of graphs with Hamilton paths and cycles. Note that if a graph has a Hamilton cycle then it also has a Hamilton path.

There are some useful conditions that imply the existence of a Hamilton cycle or path, which typically say in some form that there are many edges in the graph. An extreme example is the complete graph K_n: it has as many edges as any simple graph on n vertices can have, and it has many Hamilton cycles.

The problem for a characterization is that there are graphs with Hamilton cycles that do not have very many edges. The simplest is a cycle, C_n: this has only n edges but has a Hamilton cycle. On the other hand, figure 5.3.1 shows graphs with just a few more edges than the cycle on the same number of vertices, but without Hamilton cycles.