The Game of Thrones

and Other Graph Theory Games

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Abstract

I will be studying two impartial combinatorial games played on graphs to determine winning strategies or losing positions. One game is called Graph Nim and the other is called The Game of Thrones after the popular fantasy book and TV series. Nim is an old game dating to 16th Century Europe, which may have originated in China much earlier. Graph Nim is a variation on the original game played using multigraphs. I will attempt to extend the work of, and possibly collaborate with, M. Dufour of the University of Montreal and S. Heubach of California State University Los Angles on Circular Nim which is a special case of Graph Nim. I will also examine The Game of Thrones, which is a two player game played on a tournament. I will develop computer programs and collaborate with Larry Langley of the University of Pacific on the development of winning strategies and losing positions which we believe will lead to new structural results about tournaments.

Introduction

The game of Nim is at least centuries old, possibly originating in China, but noted in the 16th century in European countries. It consists of several stacks of tokens, and two players alternate taking one or more tokens from one of the stacks, and the player who cannot make a move loses. The formal and intense study of Nim culminated in the celebrated (at least in Game Theory circles) Sprague-Grundy Theorem[1] which is now one of the centerpieces in the theory of impartial combinatorial games. The study of impartial games typically focuses on finding a winning strategy or a losing position. In the former case, the strategy can be employed by the player who makes the first move and its use guarantees a win; in the latter, the position, or state of the game, is determined from which a player cannot win regardless of the strategy used.

We study a variation on Nim, played on a graph, which is not conducive to using the theory of Sprague-Grundy. Graph Nim is played on a multigraph instead of with stacks of tokens. On their respective turns each player selects a vertex and can delete any number of edges incident to that vertex. A winning strategy and losing position have been discovered for Graph Nim on 3 vertices. The strategy allows the first player to win unless the game starts in the losing position. The strategy for the first player is to delete the right amount of edges so that the graph is in the losing position. The losing position is when there are the same number of edges between all three pairs of vertices.

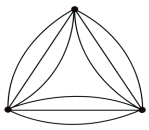

Illustration 1: The losing position for Graph Nim on 3 vertices

The Game of Thrones is a impartial combinatorial game played on a tournament. Tournaments are possibly the most well-known and heavily studied class of directed graphs. A tournament is a a directed graph in which there is an arc between every pair of vertices; that is between any two vertices

1 Sprague, R. P. (1935–36). *Über mathematische Kampfspiele*. Tohoku Mathematical Journal 41: 438–444. Grundy, P. M. (1939). *Mathematics and games*. Eureka 2: 6–8. Archived from the original on 2007-09-27. Reprinted, 1964, 27: 9–11

x and y either there is an arc from x to y (x beats y) or an arc from y to x (y beats x). A well-known application of a tournament in the sports world is known as a round robin tournament whence the "beats" terminology. In a round robin tournament each team plays every other team with no ties. The nature of a round robin tournament leads to an ambiguous interpretation of the "best" team. In an attempt to determine the dominate vertices in a tournament mathematical sociologist H. G. Landau developed the notion of a king and proved that every tournament has one[2]. A *king* in a tournament is any vertex *v* such that for every other vertex *x* the following is true: *v* beats *x* or there exists a vertex *y* such that *v* beats *y* and *y* beats *x*. There can never be exactly two kings, but otherwise there can be any number of kings in a tournament provided the number of vertices is not four[3]. Furthermore, the deletion of any vertex from a tournament leaves a tournament. These facts are the basis for the Game of Thrones.

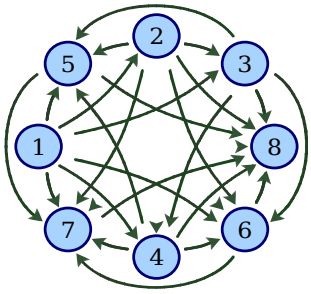

Illustration 2: A tournament with 7 vertices

The Game

The Game of Thrones was proposed by Larry J. Langley, of the University of the Pacific, and Kim A. S. Factor, of Marquette University[4]. They developed the concept of an heir. An *heir* is a vertex in a tournament that is not a king, but becomes a king when a single vertex is removed from the tournament. Langley and Factor used the concepts of heirs and kings in tournaments to develop The Game of Thrones. The game is played by two opponents, the game board is a tournament. To play the game players take turns removing vertices from the tournament. The game ends when there is only one king left in the tournament, and the winning player is the player that removed the last vertex. There are

2 H.G. Landau, *On dominance relations and the structure of animal societies*: I, Bull. Math. Biophysc. 13(1951) 1-19

3 K.B.Reid, *Every vertex a king*, Discrete Mathematics, 38 (1982) 93-98

4 L. Langley, K. Factor, *2-2 Domination or Kings and Heirs* Manuscript, 2014

many possible variations of the game, but I will focus on only two. The first game type I will examine is where a player may remove only a king from the tournament on their turn. The other type I would like to examine is where a player may remove any vertex from the tournament on their turn.

Research

I would like to develop winning strategies for Graph Nim on graphs with greater than 3 vertices. I plan to examine winning strategies and losing positions of other variations of Nim and see if I can apply them to Graph Nim. My goal in researching The Game of Thrones is to develop a winning strategy for each game type I will examine. These winning strategies will rely on the properties of tournaments, kings, and heirs; because tournaments are very diverse I will develop strategies dependent on tournament classes. The two tournament classes that I believe are most promising are Doubly Regular Tournaments and Quadratic Residue Tournaments. These classes are very structured and have many nice properties. Examining Doubly Regular Tournaments and Quadratic Residue Tournaments will require a computer program to facilitate data acquisition. I will write the required program and use it to develop winning strategies. This will allow me to learn more in the field of computer science. Much of my time will be used learning the programming language, JavaScript. I will also be required to learn more in the field of Game Theory. I will be required to gain a deeper understand of Number Theory in order to examine Quadratic Residue Tournaments. I will also have to learn more about Combinatorial Matrix Theory in order to prove conjectures that I develop. The most important skills that I will develop doing this research will be technical writing and communication skills as I prepare to present my results at conferences and write an article for a peer-reviewed journal. Some research has been done and we've found that more theory about tournaments is needed. The study of The Game of Thrones will coincidentally push the boundary of Graph Theory.

Goals

With my research on The Game of Thrones and Graph Nim I hope to develop winning strategies and losing positions for as many classes of tournaments and graphs that I can. I hope to discover enough winning strategies to write an article that I can publish in a well-known, peer-reviewed journal. Many articles concerning game strategies have been published, such as *Circular Nim Games* by M. Dufour and S. Heubach on a Nim variation called Circular Nim[5]. Dr. Brown, my faculty advisor, is in close contact with Larry Langley and Kim Factor. He also has a colleague that has done research with the game of Nim and has been in contact Dufour and Heubach. Given the popularity of impartial combinatorial games and the progress Dr. Brown and I have made so far, and that we can incorporate some structural results of Sarah Mousley from an earlier project of she and Dr. Brown's[6], we believe the prospect of publishing our results in a high quality peer-reviewed journal is good. I will also present my findings at the Mathematical Association of America Regional Conference next spring. I may also present my results at the American Mathematical Society Regional Conference in Reno, Nevada, or at the MCCC conference in Las Vegas, Nevada. I am also considering presenting my findings at the undergraduate research seminar.

Budget

The money from this SURCO grant will be used as salary for my efforts. I will work 20 hours per week for 10 weeks. The total budget requested will therefore be $2,000. The resources already present in USU's infrastructure such as computers and the Library will suffice for materials and the Math and Statistics department head, Richard Cutler has promised to provide matching funds, and additional money for travel if needed.

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5 M. Dufour, S. Heubach, *Circular Nim Games*, The Electronic Journal of Combinatorics, 20(2) (2013) 1-26.
6 Mousley, S. C., *Tournament Directed Graphs*, Honors Thesis, Utah State University, May 2013.

Time Line

I plan to begin working 20 hours per week on my research on May 12th. It is difficult to predict how fast I will discover the strategies that I am searching for, but I do plan to have enough knowledge of JavaScript to write the required program by the third week of research, so that I may begin collecting data on the Game of Thrones. I can begin working on strategies for Graph Nim immediately. I believe that I will be able to discover strategies for Graph Nim within weeks.