The Game of Thrones

A Combinatorial Game

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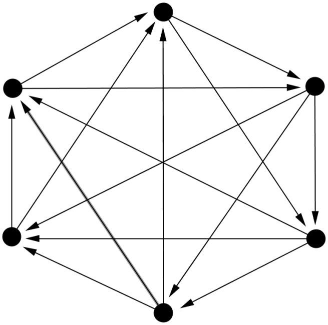
October 29, 2014

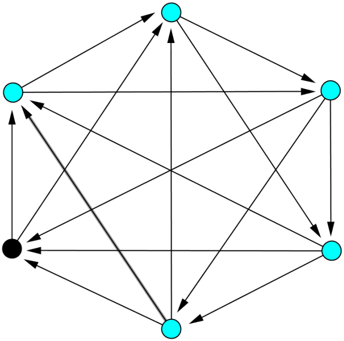
Background: Tournaments



Recall that a tournament is a complete, oriented graph.

Background: Kings

A vertex in a tournament, x, is a king if and only if for every other vertex in the tournament, y, either $x \to y$ or there exists a vertex, k, such that $x \to k$ and $k \to y$.



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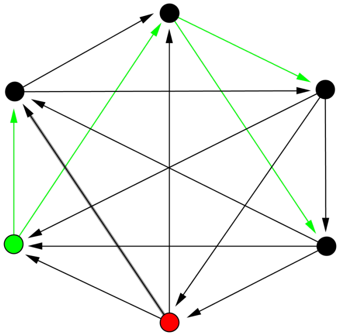

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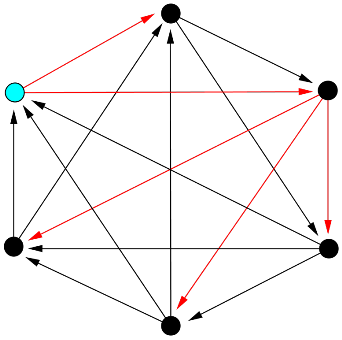


Background: Theorems

Theorem

Every Tournament has a king.

Theorem

Every induced subgraph of a tournament is also a tournament

Theorem

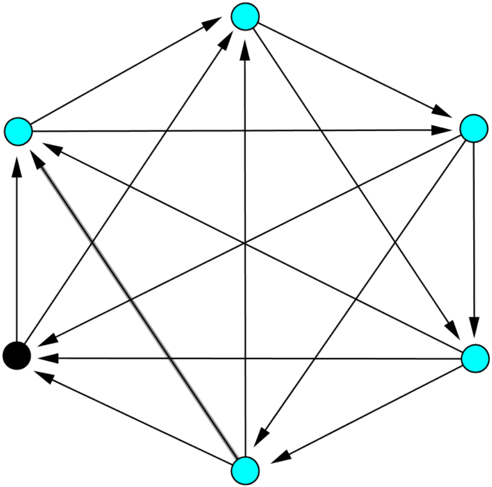
If there is exactly one king in a tournament that king is a source.

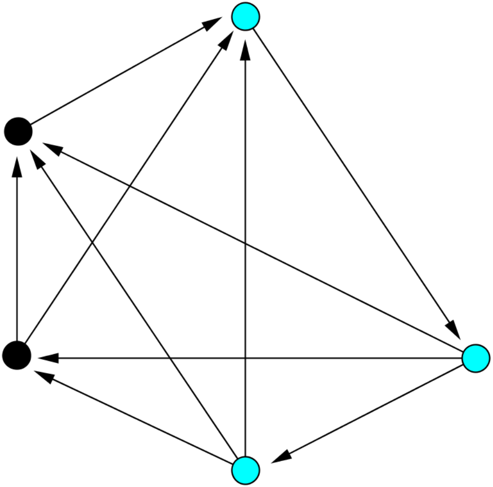
Theorem

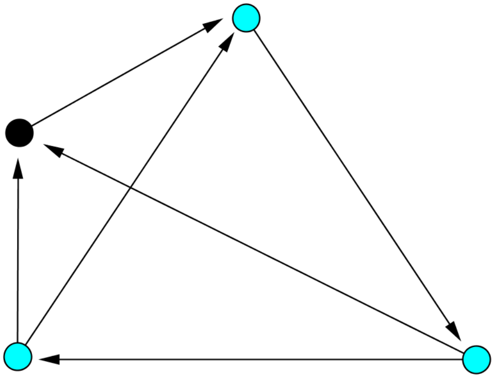
No tournament can have exactly 2 kings, and a 4-tournament can not have exactly 4 kings.

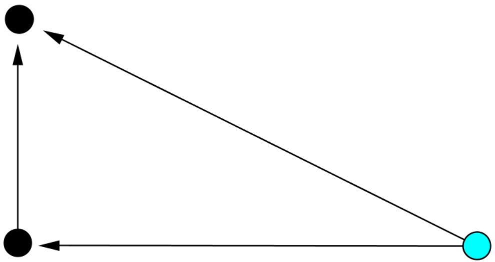

The Game of Thrones

Example Game









Gameplay: Rules

Rule 1

The game is played by two players on a tournament.

Rule 2

Players take turns deleting kings from the tournament.

Rule 3

The game ends when there is exactly one king in the tournament.

Rule 4

The last player to delete a king is the winner.

A Winning Position: Mousley's Function

Mousley's Function

$$f(m,n) = \begin{cases} 2n-2m-1, & m > \frac{n-1}{2} \\ 2m+1, & m \leq \frac{n-1}{2} \end{cases}$$

Mousley's Function allows us to determine the maximum number of vertices of score m in a tournament with n vertices.

A Winning Position: Background

Theorem

If a vertex is beaten it's beaten by a king.

Theorem

If a vertex is a source it has score $n - 1$.

A Winning Position

Theorem
If a tournament has a vertex of score $n-2$, the tournament is a winning postion.

By Mousley's Function there are at most 3 vertices of score $n-2$

$$f(n-2,n) = 2n - 2(n-2) - 1 = 3$$

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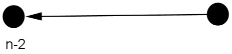

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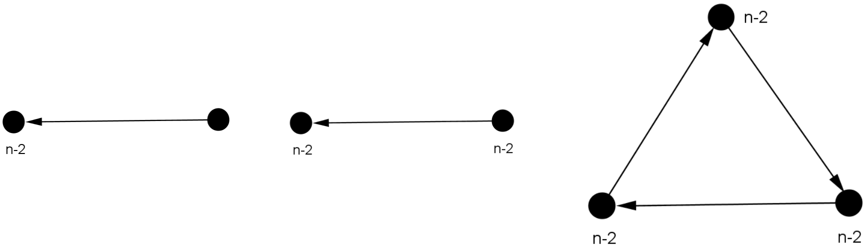

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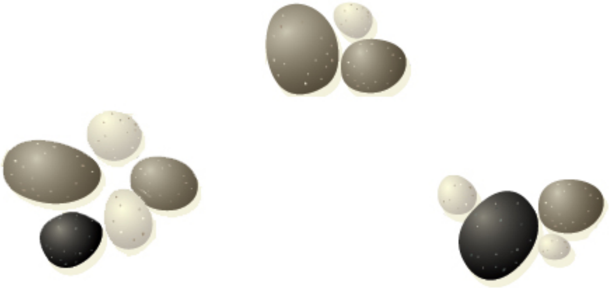
- Repeat until game is over.

The Sprauge-Grundy Theorem

Nim



The Sprauge-Grundy Theorem

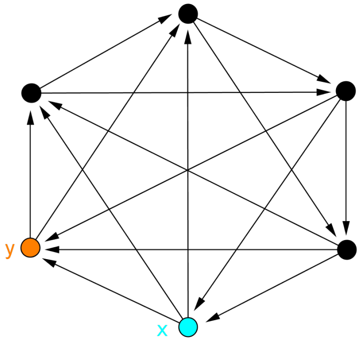
An impartial game is a game in which the allowable moves depend only on the position and not on which of the two players is currently moving.

The Sprague-Grundy theorem states that every impartial game is equivalent to a nim heap of a certain size.

Heirs

An *heir* is a vertex that is not a king, but becomes a king with the deletion of a single vertex.

If vertex y becomes a king when vertex x is deleted then y is an heir of x.



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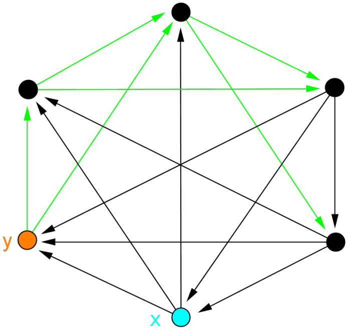

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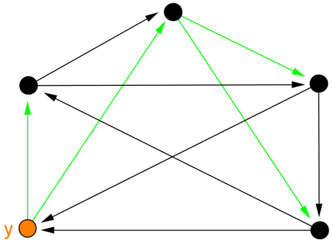

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Review

- The Game of Thrones is a two player game played on a tournament. Players take turns deleting kings, until there is exactly one king left.
- Any tournament with a vertex of score $n-2$ is a winning position.
- The Sprauge-Grundy Theorem should apply to The Game of Thrones.
- Heirs may apply to the winning strategy.

Contact Information

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