Worksheet to go along with the List of Primes worksheet.

How many primes are there less than a given quantity? We are going to derive the Prime Number Theorem π(N)~N/Ln(N)

Looking closely at the List of Primes worksheet, what do you notice about the following?

1. How many primes are there between 1 and 100? 25
2. Between 401 and 55? 17
3. Between 901 and 1000? 14
4. What do you see from this? The number of primes in any block of 100 whole numbers seems to decline.

Facts: If I continued this process up to a million, you would see that there are only 8 primes in the last hundred-block. If I took it to a trillion, there would be only 4 primes in the last hundred-block.

Do the primes eventually thin out to nothing? If I continued this list into the trillions of trillions, and trillions of trillions of trillions of trillions, would I eventually reach a point beyond which there are no more primes, so that the last prime on my list would be the last prime, the biggest prime?

No, there is no biggest prime. There will always be more primes that come.

Given the table below:

|  |  |
| --- | --- |
| N | How many primes less than N? |
| 1,000 | 168 |
| 1,000,000 | 78,498 |
| 1,000,000,000 | 50,847,534 |
| 1,000,000,000,000 | 37,607,912,018 |
| 1,000,000,000,000,000 | 29,844,570,422,660 |
| 1,000,000,000,000,000,000 | 24,739,954,287,740,860 |

Using the table above, divide the left column by the right column to get π (N), which is defined to be the number of primes up to N.

|  |  |
| --- | --- |
| N | N/ π (N) |
| 1,000 | 5.9524 |
| 1,000,000 | 12.7392 |
| 1,000,000,000 | 19.6665 |
| 1,000,000,000,000 | 26.5901 |
| 1,000,000,000,000,000 | 33.6247 |
| 1,000,000,000,000,000,000 | 40.4204 |

Look closely at the values in the table. What do you notice? — Each value goes up by about 7 each time. It wobbles between 6.8-7.0.

Now, we are going to add two more columns to our table. In the Ln(N) column, take the Ln(N) and write the value down. For the % error column, take the Ln(N)/(N/π(N))-1.

|  |  |  |  |
| --- | --- | --- | --- |
| N | N/ π (N) | Ln(N) | % error |
| 1,000 | 5.9524 | 6.9077 | 16.0490 |
| 1,000,000 | 12.7392 | 13.8155 | 8.4487 |
| 1,000,000,000 | 19.6665 | 20.7232 | 5.3731 |
| 1,000,000,000,000 | 26.5901 | 27.6310 | 3.9146 |
| 1,000,000,000,000,000 | 33.6247 | 34.5378 | 2.7156 |
| 1,000,000,000,000,000,000 | 40.4204 | 41.4465 | 2.5386 |

What can you conclude by looking at the values of N/ π (N) compared to Ln(N)? – The following statement seems reasonable: N/ π (N) is close to Ln(N); and the larger N gets, the closer (proportionally) it gets.

How could we write what you stated above into an expression? – N/ π (N)~Ln(N), this is pronounced “N over pi of N tends asymptotically to log N.” Or rearranging the equation, we get, π (N)~N/Ln(N).

You have just derived the Prime Number Theorem.