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EVERY VERTEX A KING

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A king in a tournament is a vertex which can reach every other vertex via a 1-path or 2-path. A new inductive proof is given for the existence of an n-tournament with exactly k kings for all integers $n \geq k \geq 1$ with the following exceptions: $k = 2$ with n arbitrary, and $n = k = 4$ (in which cases no such n-tournament exists). Also, given an n-tournament T, the smallest order m is determined so that there exists an m-tournament W which contains T as a subtournament and so that every vertex of W is a king. Bounds are obtained in a similar problem in which the kings of W are exactly the vertices of T.

In a delightful exposition on the use of tournaments to model dominance in flocks of chickens, S.B. Maurer [3] defined a *king* in a tournament T as a vertex x in T such that for every other vertex y in T, either x dominates y in T or T contains a 2-path from x to y. He proved that for all integers $n \geq k \geq 1$ there exists an n-tournament with exactly k kings with the following exceptions: $k = 2$ with n arbitrary, and $n = k = 4$ (in which cases no such n-tournament exists). The fact that no tournament has exactly two kings appears implicitly in a problem posed by D.L. Silverman [8] and solved by J.W. Moon [4] and occurs in the treatment of tournaments by F. Harary *et al.* [1]. The idea to use kings in the study of dominance in tournaments emerged from work by the mathematical sociologist H.G. Landau [2] who proved that every vertex of maximum score is a king. The purpose of this article is to answer several questions on kings posed by Maurer [3].

If x is a vertex in a tournament, then $O(x)$ will denote the out-set of x, that is, the set of vertices dominated by x. The cardinality of $O(x)$ will be denoted by $d^+(x)$. Similarly, $I(x)$ will denote the in-set of x, those vertices dominating x, and its cardinality will be denoted by $d^-(x)$. A tournament in which every vertex is a king will be called an *all-kings tournament*. For terminology and notation not introduced here the reader is referred to the monograph by J.W. Moon [5] or the recent survey article by L.W. Beineke and the author [6].

The first result is an inductive construction (solving Problem 4 in [3]) to be used in a new proof of a result by Maurer [3, Theorems 6 and 11].

Lemma 1. *If there exists an all-kings n-tournament, $n \geq 4$, then there exists an all-kings $(n+1)$-tournament.*

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Proof. Let T be an all-kings n-tournament, $n \geq 4$. Let x be a vertex of T for which $d^+(x) \geq d^-(x)$; thus, $d^+(x) \geq 2$ since $n \geq 4$. Let W be the $(n+1)$-tournament obtained from T by adjoining to T a new vertex y such that y dominates x and each vertex in $I(x)$ and y is dominated by each vertex in $O(x)$. Let Z denote those vertices in $I(x)$ which dominate no vertex in $O(x)$. If Z is empty, then W is an all-kings $(n+1)$-tournament. If Z is not empty, then $d^-(x) - |Z| < d^+(x)$ by choice of x. That implies that there exists a vertex w in $O(x)$ so that for every arc zv with z in Z and v in $I(x) - Z$ there exists a 2-path from v to z in $T - w$. Form W' from W by reversing all the arcs between Z and w. Note that x dominates some vertex different from w. Then W' is an all-kings $(n+1)$-tournament. This completes the proof.

The previous lemma is now used to give a new proof of the following result.

Theorem 2 [3]. *There exists an n-tournament with exactly k kings for all integers $n \geq k \geq 1$, with the following exceptions: $k = 2$ with n arbitrary, and $n = k = 4$.*

Proof. If $k \neq 4$ and there exists an all-kings k-tournament T, then, as in [3], an n-tournament with exactly k kings can be obtained from T by adjoining $n - k$ new vertices, each of which is dominated by every vertex in T and dominance among the $n - k$ new vertices is arbitrary. None of the four 4-tournaments has 4 kings, but a similar construction suffices if $k = 4$ and $n \geq 5$. It is based on the existence of a 5-tournament with exactly 4 kings (as in [3], such a tournament can be obtained by adding a fifth vertex x to the strong 4-tournament in which y and z are the vertices of score 2 and y dominates z, where x dominates exactly z). Also, no tournament has exactly two kings, for if x and y are two kings in a tournament and x dominates y, then any vertex of maximum score in the subtournament with vertex set $I(x)$ is a king of that subtournament (by Landau's result), hence a king in the whole tournament.

Thus, it is sufficient to treat the case $n = k \neq 2, 4$. For each odd integer $2m + 1$, $m \geq 0$, there exists a regular $(2m+1)$-tournament (i.e., each vertex has score m), hence each of its vertices is a king by Landau's result. For example, the rotational tournaments and, in case $2m+1$ is an odd power of a prime congruent to 3 (mod 4), the quadratic residue tournaments are regular tournaments (see [6, p. 172]). By Lemma 1, the proof is complete.

The directional dual of a king is called a serf [3]. That is, a vertex is a *serf* if it can be reached by every other vertex via a 1-path or 2-path. Thus, x is a king of a tournament T if and only if x is a serf of $\bar{T}$, the converse of T. By duality and Theorem 2, there exists an n-tournament with exactly s serfs for all integers $n \geq s \geq 1$, with the following exceptions: $s = 2$ with n arbitrary, and $n = s = 4$.

Maurer (Problem 1 in [3]) asked to determine all 4-tuples (n, k, s, b) for which there exists an n-tournament with exactly k kings and s serfs such that exactly b

of the kings are also serfs. Such a tournament is called a (n, k, s, b)-tournament. By directional duality, there is no loss of generality in assuming $k \geqslant s$. Clearly it is necessary that $n \geqslant k \geqslant s \geqslant b \geqslant 0$ and $n \geqslant k+s-b$. The characterization of such 4-tuples is given in the next theorem, the proof of which is too long to include here.

Theorem 3 [7, Theorem 12]. *Suppose that $n \geqslant k \geqslant s \geqslant b \geqslant 0$ and $n > 0$. There exists a (n, k, s, b)-tournament if and only if the following conditions hold:*

(1) $n \geqslant k+s-b$,

(2) $s \neq 2$ *and* $k \neq 2$,

(3) *either* $n = k = s = b \neq 4$ *or* $n > k$ *and* $s > b$,

(4) (n, k, s, b) *is none of* $(n, 4, 3, 2)$, $(5, 4, 1, 0)$, *or* $(7, 6, 3, 2)$.

Maurer (Problem 2 in [3]) also asked to determine those n-tournaments T which are contained in a tournament whose kings are exactly the vertices of T. Subsequently he provided a characterization which is given in the following theorem.

Theorem 4. *A nontrivial n-tournament T is contined in a tournament whose kings are the vertices of T if and only if T contains no transmitter.*

Proof. The necessity follows from the fact that any vertex x that is dominated by some other vertex must be dominated by a king. For, any king in the subtournament with vertex set $I(x)$ is such a king.

On the other hand, if T is nontrivial and has no transmitter, then $n \geqslant 3$. Let $x_1, \ldots, x_n$ denote the vertices of T, and let T' denote an isomorphic, disjoint copy of T with vertices $x_1', \ldots, x_n'$ where x_i' corresponds to x_i, $1 \leqslant 1 \leqslant n$. Form W from T and T' where each vertex of T dominates each vertex of T' except x_i' dominates x_i, $1 \leqslant i \leqslant n$. Then each x_i is a king of W, but no x_i' is a king of W since x_i is not a transmitter of T, $1 \leqslant i \leqslant n$. The proof is complete.

Note that in the proof of the sufficiency the order of W is $2n$ for $n \geqslant 3$, but this is not best possible in the sense that a W of smaller order can be obtained. For example, if T is the strong 4-tournament, then it is easy to find a 5-tournament W so that the kings of W are exactly the vertices of T.

If T is a nontrivial tournament without a transmitter, let $m(T)$ denote the least order of a tournament which contains T and whose kings are exactly the vertices of T. By the preceding remarks, $m(T) = 5$ for the strong 4-tournament T.

In order to discuss a lower bound on $m(T)$ (for non-trivial T without a transmitter), some notation will be needed. Let T be any n-tournament with vertex set V. Let V_1 denote the set of kings of T, and, inductively, let $V_i = \{x \in V-(V_1 \cup \cdots \cup V_{i-1}) \mid$ for each $y \in V-(V_1 \cup \cdots \cup V_{i-1})$, $y \neq x$, there is a 1-path or 2-path in T from x to $y\}$, $2 \leqslant i \leqslant n$. Let $p = p(T)$ denote the largest index such that V_p is nonempty. Then V_i is nonempty for $1 \leq i \leq p$, and $p = 1$ if and only if each vertex of T is a king. For example, if T is the transitive n-tournament, then $p(T) = n$ as each V_i is a singleton.

Lemma 5. *Let T be an n-tournament, and let V_i, $1 \leq i \leq p = p(T)$, be as above. Then for each $i, 2 \leq i \leq p$, and for each v_i in V_i there exist vertices v_j in V_j, $1 \leq j \leq i-1$, such that $O(v_i) \subseteq O(v_{i-1}) \subseteq \cdots \subseteq O(v_1)$.*

The proof is a straightforward induction on i, $2 \leq i \leq p$, so it will be omitted.

In the following if T is a tournament, then $\{\log_2 p(T)\}$ will be denoted by $l(T)$, or by l if no confusion will result, where $\{x\}$ denotes the smallest integer greater than or equal to x.

The next theorem not only yields a lower bound for the order of the tournament whose existence is guaranteed by Theorem 4, but also yields a lower bound in a similar problem to be subsequently treated.

Theorem 6. *If T is an n-tournament which is a subtournament of an m-tournament W and each vertex of T is a king of W, then $m \geq n + l(T)$.*

Proof. The result clearly holds if $p(T) = 1$. So, suppose that T and W are as in the statement of the theorem and that $p = p(T) > 1$. Let v_p be a vertex in V_p, and let $v_1, \ldots, v_{p-1}$ be as in Lemma 5. Denote the vertex set of T by V. If v_i and v_j dominate the same subset of vertices in $W - B$, for some i and j, $1 \leq i < j \leq p$, then no vertex in $W - V$ is used in any 1-path or 2-path from v_j to v_i. But, v_j is a king in W, so there exists a vertex z in V such that z is in $O(v_j)$ and v_i is in $O(z)$. But, by Lemma 5, z is in $O(v_i)$, contradicting the asymmetry of T. Thus, no two v_i dominate the same subset of vertices in $W - V$, and $p \leq 2^{m-n}$. Thus, $m \geq n + l(T)$.

Corollary 7. *If T is a nontrivial n-tournament without a transmitter, then $n + l(T) \leq m(T) \leq 2n$.*

Problem. Given an n-tournament T without a transmitter, improve the bounds in Corollary 7 on $m(T)$.

A problem similar to the previous one is to characterize those n-tournaments T for which there exists an all-kings m-tournament W such that T is a subtournament of W. In fact, every n-tournament T has this property. For, if $p(T) = 1$, then take $W = T$. If $p(T) > 1$, there exists an $(n+1)$-tournament T' which contains T as a subtournament and $p(T') < p(T)$. T' can be obtained from T by adjoining a new vertex z so that z is dominated by exactly those vertices in $V_2 \cup U_1$, where $U_1 = \{x \in V_1 \mid x$ is dominated by every vertex in $V_2\}$ and $V_1, V_2, \ldots, V_p$ are as in the definition of $p(T)$. Then Lemma 5 implies that each vertex of V_2 is dominated by some vertex of V_1, so that the set of kings of T' is $V_1 \cup V_2 \cup \{z\}$ and $p(T') < p(T)$. By repeating this construction, if necessary, W is obtained. Of course, $m \geq n + l(T)$ by Theorem 6. In fact, this lower bound can be achieved in the present problem by suitably modifying the construction just given.

Theorem 8. *Let T be an n-tournament. Then the least order of an all-kings tournament which contains T is $n + l(T)$.*

Proof. The proof will proceed by induction on $l(T)$. Note that $l(T) = 0$ if and only if $p(T) = 1$, in which case T itself is an all-kings tournament. Assume that the theorem is true for tournaments Z for which $l(T) < k$, where $k \geq 1$, and suppose that T is an n-tournament for which $l(Z) = k$. Let $V_i = V_i(T)$, $1 \leq i \leq p(T)$, be as in the definition of $p = p(T)$ above, where the functional notation will be used when needed to emphasize the dependence on the tournament under consideration. Let $U_i = \{x \in V_i \mid x \text{ is dominated by every vertex in } V_{i+1}\}$, $1 \leq i \leq p - 1$. Then $U_i \neq V_i$ by definition of V_{i+1}, $1 \leq i \leq p - 1$. Note that no vertex in U_i dominates every vertex in $V_i - U_i$, as otherwise each vertex in V_{i+1} can reach each vertex in V_i via a 1-path or a 2-path, a contradiction to the definition of V_{i+1}, $1 \leq i \leq p - 1$.

Form an $(n+1)$-tournament T_1 by adjoining to T a new vertex z such that z dominates exactly the vertices in

$$\bigcup \{V_i - U_i \mid 1 \leq i \leq p - 1, i \text{ odd}\} \cup W,$$

where

$$W = \begin{cases} V_p, & \text{if } p \text{ is odd} \\ \emptyset, & \text{if } p \text{ is even.} \end{cases}$$

Now, z is a king of T_1. For let x be any vertex of T which is not dominated by z. If x is in U_i for some odd i, $1 \leq i \leq p - 1$, then as noted above, there is a vertex in $V_i - U_i$ on a 2-path from z to x. If x is in V_i for some even i, $2 \leq i \leq p$, then as no vertex of V_i contains every vertex in V_{i-1}, there is some vertex in $V_{i-1} - U_{i-1}$ which is on a 2-path from z to x.

In addition, each vertex of V_1 is a king of T_1. For, each vertex of V_1 is a king of T, each vertex of $U_1 \cup V_2$ dominates z, and by the definition of U_1, from each vertex of $V_1 - U_1$ there is a 2-path to z which includes a vertex of V_2.

Note that from each vertex of V_i, i even, to each vertex of $V_{i-1} - U_{i-1}$ there is a 2-path using z, $2 \leq i \leq p$, and each vertex of V_i, i even, dominates every vertex of U_{i-1}, $2 \leq i \leq p$. In particular, each vertex of V_2 is a king of T_1.

Consequently, $\{z\} \cup V_1(T) \cup V_2(T) \subseteq V_1(T_1)$. If $p(T) = 2$, clearly these two sets are equal. If $p(T) > 2$, then these two sets are equal by Lemma 5. Moreover, by the previous paragraph and Lemma 5, if $p(T) > 2$, then

$$V_j(T_1) = V_{2j-1}(T) \cup V_{2j}(T), \quad 1 \leq j \leq \{\tfrac{1}{2}(p - 1)\},$$

and if p is odd, $V_{(p+1)/2}(T_1) = V_p(T)$. Hence,

$$p(T_1) = \{\tfrac{1}{2}p(T)\} \quad \text{and} \quad l(T_1) = k - 1.$$

By the induction hypothesis there exists an all-kings m-tournament W such that T_1 is a subtournament of W and

$$m=(n+1)+l(T_1).$$

So T is a subtournament of W and

$$m=(n+1)+(k-1)=n+l(T).$$

By induction and Theorem 6, the result follows.

The worst case in the previous theorem occurs when T is transitive.

Theorem 9. *The smallest order of an all-kings tournament which contains the transitive n-tournament is* $2n$.

Remark. If T has no transmitter, then it may be possible to add 'a few' vertices to W constructed in the proof of Theorem 8 to obtain an all-kings tournament, thus improving the upper bound in Corollary 7. Some facts about W that might prove useful are: $V_p \cup U_{p-1}$ dominates all of $z_1, \ldots, z_k$; the subtournament of W with vertex set $\{z_1, \ldots, z_k\}$ is transitive and z_j dominates z_i if and only if $1 \leq i < j \leq k$; every 1-path or 2-path in W from any vertex of V to any vertex of $V_p \cup U_{p-1}$ lies entirely in T.

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